

ITERATED MONODROMY GROUPS OF INTERMEDIATE GROWTH

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ABSTRACT. We give new examples of groups of intermediate growth, by a method that was first used by Bux and Pérez. Our examples are the groups generated by the automata with the kneading sequences $11(0)^\omega$ and $0(011)^\omega$. By results of Nekrashevych, both of these groups are iterated monodromy groups of complex post-critically finite quadratic polynomials.

We include a complete, systematic description of the Bux-Pérez method. We also prove, as a sample application of the method, that the groups determined by the automata with kneading sequence $1(0^k)^\omega$ ($k \geq 2$) have intermediate growth, although this result is implicit in a survey article by Bartholdi, Grigorchuk, and Sunik.

The paper concludes with an example of a group with no admissible length function; i.e., the group in question admits no length function with the properties required by the Bux-Pérez method. Whether the group has intermediate growth appears to be an open question.

1. INTRODUCTION

Let $p : \mathbb{C} \rightarrow \mathbb{C}$ be a complex polynomial. We say that p is *post-critically finite* if, for each critical point c , the set of all forward iterates $\{p(p \dots (c))\}$ of c is a finite set. Nekrashevych [6] has shown how to associate a group, called an *iterated monodromy group*, to any post-critically finite complex polynomial. The iterated monodromy group of p , denoted $\text{IMG}(p)$, acts on an infinite rooted n -ary tree if the degree of p is n .

We first began this project because of our interest in the following conjecture from [2], where it is attributed to Nekrashevych:

Conjecture 1.1. *If $p : \mathbb{C} \rightarrow \mathbb{C}$ is a post-critically finite quadratic polynomial with pre-periodic kneading sequence, then $\text{IMG}(p)$ has intermediate growth.*

The first positive evidence was obtained by Bux and Pérez [2], who showed that $\text{IMG}(z^2 + i)$ has subexponential growth. (The proof that $\text{IMG}(z^2 + i)$ also has superpolynomial growth is comparatively straightforward, so their work proves that $\text{IMG}(z^2 + i)$ has intermediate growth.) There are known counterexamples, however. Grigorchuk and Zuk [5] showed that $\text{IMG}(z^2 - 1)$ has exponential growth. The tuning [3] of $z^2 - 1$ by $z^2 + i$ results in a post-critically finite quadratic polynomial $g(z) = z^2 + c$ with pre-periodic kneading sequence such that $\text{IMG}(z^2 - 1)$ embeds in $\text{IMG}(g)$. It follows easily that $\text{IMG}(g)$ also has exponential growth, making it a counterexample to Conjecture 1.1. (Note that $z^2 - 1$ has a periodic kneading

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sequence, so it is not a counterexample in itself.) The following conjecture appears to be open:

Conjecture 1.2. *If $p : \mathbb{C} \rightarrow \mathbb{C}$ is a non-renormalizable post-critically finite quadratic polynomial with pre-periodic kneading sequence, then $\text{IMG}(p)$ has intermediate growth.*

The hypothesis of non-renormalizability rules out the counterexamples to Conjecture 1.1 that arise from tuning.

Our goal here is to give two more examples in support of the latter conjecture, namely the iterated monodromy groups of polynomials with the kneading sequences $11(0)^\omega$ and $0(011)^\omega$. (The kneading sequence of a kneading automaton is described in Definition 2.15; Theorem 2.21 says that the latter definition agrees with the classical definition of the kneading sequence of a polynomial up to relabeling.) We also give a short proof that the groups generated by the automata with the kneading sequences of the form $1(0^k)^\omega$ have intermediate growth, although a proof that these groups have intermediate growth can be obtained from Theorem 10.5 of [1]. A secondary goal is to provide an exposition of the Bux-Pérez method. We attempt to isolate the precise hypotheses that are necessary to make their arguments work, and state general theorems. The main result in this direction is Theorem 3.23, which gives a simple sufficient condition for the group of a kneading automaton over an alphabet with two letters to have subexponential growth.

The paper is structured as follows. In Section 2, we review the definition of automata, and explain how an automaton can be used to define a group that acts by automorphisms on a rooted tree. Section 3 contains an exposition of the Bux-Pérez method. Section 4 contains proofs that the groups determined by the automata with the kneading sequences $1(0^k)^\omega$ ($k \geq 1$), $11(0)^\omega$, and $0(011)^\omega$ have intermediate growth. In Section 5, we give an example of a group defined by an automaton with pre-periodic kneading sequence to which the methods of Bux and Pérez do not apply. (Specifically, the group in question has no admissible length function – see Definition 3.9.)

All of the results in Sections 4 and 5 were proved by the authors at SUMSRI, an REU program based at Miami University, during the summer of 2011. The authors also produced an article as part of the REU, which can be found at www.units.muohio.edu/sumsri/sumj/2011/fp_alg.pdf. The material in Section 3 was contributed by Daniel Farley.

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2. BACKGROUND

Essentially all of the material in this section has appeared in [6]. We gather it here (sometimes in slightly altered form) for the reader's convenience.

2.1. Automata, Moore Diagrams, Trees.

Definition 2.1. Let X be an alphabet. An *automaton* A over X is given by:

- (1) a set of *states*, usually also denoted A ;
- (2) a map $\tau : A \times X \rightarrow X \times A$.

For each state $a \in A$, we define a function $\tau_a : X \rightarrow X$ by the rule $\tau_a(x) = \pi_1(\tau(a, x))$, where $\pi_1 : X \times A \rightarrow X$ is projection on the first coordinate. If $\tau_a \neq \text{id}_X$,

we say that a is an *active state*. We say that the automaton A is *invertible* if each $\tau_a : X \rightarrow X$ is a bijection.

Automata can be conveniently described using Moore diagrams.

Definition 2.2. Let A be an automaton over the alphabet X . The *Moore diagram* for A is a directed labelled graph Γ , defined as follows. The vertices of Γ are the states of A . If $a, b \in A$ and $\tau(a, x) = (y, b)$, then there is a directed edge from a to b with the label (x, y) .

Example 2.3. We define an automaton A as follows. The states are a, b, t , and id . The alphabet is $X = \{0, 1\}$. We define the function $\tau : A \times X \rightarrow X \times A$ by the rule:

$$\begin{array}{ll} \tau(a, 0) = (0, id) & \tau(b, 0) = (0, b) \\ \tau(a, 1) = (1, t) & \tau(b, 1) = (1, a) \\ \tau(t, 0) = (1, id) & \tau(id, 0) = (0, id) \\ \tau(t, 1) = (0, id) & \tau(id, 1) = (1, id) \end{array}$$

The Moore diagram for this automaton is pictured in Figure 1.

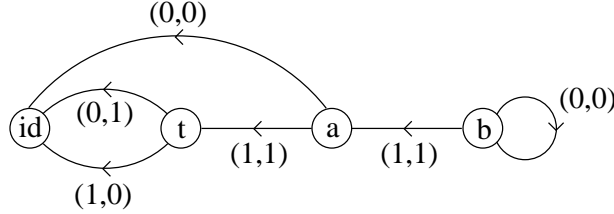


FIGURE 1. The Moore diagram for the automaton from Example 2.3.

Definition 2.4. Let X be an alphabet. We let X^* denote the free monoid generated by X . Thus, X^* is the collection of all positive strings in the letters of X , including the empty string. We can associate to X^* a tree, which we also denote X^* , as follows: the vertices of the tree are the members of X^* , and there is an edge connecting two vertices $w_1, w_2 \in X^*$ if and only if $w_2 = w_1x$, for some $x \in X$ (or $w_1 = w_2x$, for some $x \in X$).

It is easy to see that if $|X| = n$, then X^* is a complete rooted n -ary tree, i.e., there is a root vertex, \emptyset , of degree n , and all other vertices have degree $n + 1$.

2.2. The group determined by an automaton. We will now explain how an invertible automaton over X determines a group of automorphisms of the tree X^* .

Definition 2.5. Let A be an automaton over X . Let $a \in A$ and $x \in X$. We define $a|_x = \pi_2 \tau(a, x)$, where $\pi_2 : X \times A \rightarrow A$ is projection on the second factor.

For each $a \in A$, we can define a function $a : X^* \rightarrow X^*$ by induction on the length of a word $w \in X^*$. If $|w| = 0$ (so w is the null string), then we set $a(w) = w$. If $|w| > 0$, then we can write $w = xw_1$, for some $x \in X$ and $w_1 \in X^*$. We define $a(w) = \tau_a(x)a|_x(w_1)$.

It is easy to see that the function $a : X^* \rightarrow X^*$ preserves parents and children: if $w_2 = w_1x$ for some $x \in X$ (so w_2 is a child of w_1), then $a(w_2)$ is also a child of $a(w_1)$. If the automaton A is invertible then $a : X^* \rightarrow X^*$ is an automorphism for each $a \in A$ ([6]; pg. 7).

Definition 2.6. Let A be an invertible automaton. The group defined by A , $G(A)$, is the subgroup of $\text{Aut}(X^*)$ generated by:

$$\{a : X^* \rightarrow X^* \mid a \in A\}.$$

While our definition of $a|_x$ (for $a \in A$ and $x \in X$) is good enough to define the action of $G(A)$ on its associated tree, we will often need a definition of $w|_x$, where $w \in A^*$ and $x \in X$.

Definition 2.7. Let $a_1a_2 \dots a_n \in A^*$. For $x \in X$, we define $(a_1a_2 \dots a_n)|_x$ in A^* by the rule:

$$(a_1a_2 \dots a_n)|_x = a_1|_{\tau_{a_1}\tau_{a_2} \dots \tau_{a_n}(x)} \dots a_{n-1}|_{\tau_{a_n}(x)} a_n|_x.$$

Note 2.8. An easy way to compute $(a_1a_2 \dots a_n)|_x$ is as follows. Form the concatenation $a_1a_2 \dots a_nx$ (in $(A \cup X)^*$) and then regard the function $\tau : A \times X \rightarrow X \times A$ as a rule telling us to replace a substring $\widehat{a}\widehat{x}$ ($\widehat{a} \in A$, $\widehat{x} \in X$) by the string $\widetilde{x}\widetilde{a}$, where $\tau(\widehat{a}, \widehat{x}) = (\widetilde{x}, \widetilde{a})$. Once we have rewritten the original word $a_1a_2 \dots a_nx$ in the form $y\widehat{a}_1 \dots \widehat{a}_n$, where $y \in X$, it follows that $(a_1 \dots a_n)|_x = \widehat{a}_1 \dots \widehat{a}_n$.

For instance, if A is the automaton from Example 2.3, then

$$abta \cdot 1 = abt1t = ab0(id)t = a0b(id)t = 0(id)b(id)t.$$

We omit occurrences of the identity state (Definition 2.12) from $(a_1a_2 \dots a_n)|_x$. (Note also that $(a_1a_2 \dots a_n)|_x$ is a string in $(A - \{id\})^*$, not an element of a group – this distinction is important in Definition 2.14.) It follows that $abta|_1 = bt$.

Definition 2.9. Let G be a group defined by an automaton A . We let G_n denote the n th level stabilizer, defined as follows:

$$G_n = \{g \in G \mid g \cdot w = w, \text{ for all } w \in X^* \text{ such that } |w| \leq n\}.$$

Definition 2.10. Let g_0, g_1, \dots, g_{n-1} be automorphisms of the tree X^* , where $|X| = n$. Assume, without loss of generality, that $X = \{0, 1, \dots, n-1\}$. We let $g = (g_0, g_1, \dots, g_{n-1})$ be the automorphism defined by the rule $g(jw) = jg_j(w)$, for $j \in X$ and $w \in X^*$. (Thus, g fixes the top level of the tree and acts like the automorphism g_j on the j th branch from the root.)

Note that if G is a group defined by an automaton A over $X = \{0, 1, \dots, n-1\}$, and $g \in G_1$ can be represented by a word in $(A - \{id\})^*$, then $g = (g|_0, \dots, g|_{n-1})$. We will make extensive use of this fact in subsequent sections.

We will always work with reduced automata.

Definition 2.11. An automaton A is *reduced* if different states a of A induce different functions $a : X^* \rightarrow X^*$.

Any automaton can be reduced, i.e., there is an algorithm which finds a reduced automaton whose states define the same set of functions $a : X^* \rightarrow X^*$ as the given automaton ([6]; pg. 8).

Definition 2.12. A state a is called an *identity state* if $a : X^* \rightarrow X^*$ is the identity automorphism.

It is clear from the definition that a reduced automaton can have at most one identity state.

2.3. Kneading automata of quadratic polynomials. Throughout this section, we assume that X is a two-letter alphabet. The definitions in this section are all drawn from [6]. In most cases, our definitions look simpler than the ones in [6] because we are restricting our attention to a two-letter alphabet, while the discussion in [6] is more general. Note in particular that Definition 2.13 would be an incorrect definition of kneading automata if we replaced X with a larger set.

Definition 2.13. ([6]; pg. 167) Let A be an invertible reduced automaton over $X = \{0, 1\}$. We say that A is a *kneading automaton* if

- (1) there is only one active state;
- (2) in the Moore diagram of A , each non-identity state has exactly one incoming arrow;
- (3) at most one outgoing arrow from the active state leads to a non-identity state.

Definition 2.14. ([6]; pg. 177) A kneading automaton A over $X = \{0, 1\}$ is *planar* if there is some linear ordering $a_1 \dots a_m$ of the non-trivial states of A such that $((a_1 \dots a_m)^2)_{|x}$ is a cyclic shift of $a_1 \dots a_m$ for each $x \in X$.

If A is a kneading automaton over the two-letter alphabet X , then there are two general forms that A might take. Consider the result of deleting the identity state and all arrows in the Moore diagram for A that lead to the identity state. It is not too difficult to see that the resulting directed graph Γ_A is topologically either a circle, or a circle with a sticker $([0, 1])$ attached at one of its ends. (In the latter case, the active state of A is the unique vertex of degree 1.) It is also clearly possible to reconstruct the Moore diagram of A from Γ_A (since all of the arrows that are missing from Γ_A must lead to the identity state).

Definition 2.15. ([6]; pg. 183) Let A be a kneading automaton. We define the *kneading sequence* of A as follows.

If Γ_A is topologically a circle with m edges, then the kneading sequence for A has the form $(\ell_1 \ell_2 \dots \ell_m)^\omega$, where ℓ_1 is the label of the (unique) arrow leading from a_1 (say) into the active state, ℓ_2 is the label of the arrow leading from a_2 into a_1 , and so forth, so that ℓ_m leads from the active state into a_{m-1} . (In other words, we trace the arrows backwards from the active state, while recording the labels in the order that they are encountered. When we reach the active state again, having recorded the string $\ell_1 \ell_2 \dots \ell_m$, we define the kneading sequence to be $(\ell_1 \dots \ell_m)^\omega$.)

If Γ_A is topologically a circle with a sticker, then we similarly trace the arrows backward from the active state (which is necessarily the unique vertex of degree 1 in Γ_A) and record the labels. The kneading sequence takes the form $u(v)^\omega$, where u is the (non-empty) label of the sticker, and v is the label of the circle. The latter label v is read from Γ_A in essentially the same way as before.

We abbreviate the four possible labels $(0, 0)$, $(1, 1)$, $(0, 1)$, and $(1, 0)$ by 0 , 1 , $*_0$, and $*_1$ (respectively).

Example 2.16. The automaton A in Example 2.3 is a kneading automaton. Its associated graph Γ_A is topologically a circle with a sticker. If we trace the arrows backwards from t to a , then to b , and then to b again, we read the labels $(1, 1)$, $(1, 1)$, and $(0, 0)$ (respectively). Since following the arrow backwards from b leads to

b , the kneading sequence repeats after this. It follows that the kneading sequence is $11(0)^\omega$.

Note 2.17. It is straightforward to check that a kneading automaton can be recovered from its kneading sequence. Note also that the automaton with the kneading sequence $1(0)^\omega$ is different from the automaton with the kneading sequence $1(00)^\omega$ (for example).

Definition 2.18. ([6]; pg. 184) We say that a kneading sequence is *pre-periodic* if it has the form $u(v)^\omega$, where $u, v \in X^*$ are non-trivial strings. (That is, if the graph Γ_A is topologically a circle with a sticker attached.)

We use a characterization of “bad isotropy groups” on page 184 of [6] as a definition:

Definition 2.19. A kneading automaton A over $X = \{0, 1\}$ has *bad isotropy groups* if and only if its kneading sequence is pre-periodic and the word v from Definition 2.18 is a proper power.

Definition 2.20. A complex polynomial $p : \mathbb{C} \rightarrow \mathbb{C}$ is called *post-critically finite* if for each critical point c (i.e., $p'(c) = 0$), the set

$$\{p(p(\dots p(c)))\}$$

is finite.

Theorem 2.21. *Let A be an invertible reduced kneading automaton over X , where $|X| = 2$. If A is planar and does not have bad isotropy groups, then $G(A)$ is the iterated monodromy group of a post-critically finite quadratic polynomial $p : \mathbb{C} \rightarrow \mathbb{C}$, and the kneading sequence of A is also the kneading sequence of p (up to relabeling of X).*

Proof. This follows from Nekrashevych’s Theorem 6.9.6 [6] and the discussion on page 187 of [6]. \square

3. THE BUX-PÉREZ METHOD

Let A be a kneading automaton with pre-periodic kneading sequence. It frequently happens that an element $g \in G(A)_1 = G_1$ becomes shorter in total length when it is written as an ordered pair, i.e., as $g = (g_{|0}, g_{|1})$. The arguments of Bux and Pérez [2] show that if some positive proportion of group elements g possess this property, then G will have subexponential growth.

The goal of this section is to give a careful statement and proof of this fact. Our approach is based on [2].

3.1. Length Functions, rewriting rules and weak reduced forms. We assume, from now on, that A is a kneading automaton with a pre-periodic kneading sequence, and $X = \{0, 1\}$. We let t denote the (unique) active state of A . Our assumptions imply that $t : X^* \rightarrow X^*$ is defined by the rule $t(0w) = 1w$; $t(1w) = 0w$.

We let S be the set of non-trivial, non-active states of A . We set $H = \langle S \rangle \leq G(A)$. We will frequently write G in place of $G(A)$ when the automaton A is understood.

It is straightforward to check that every generator $a \in A - \{id\}$ has order 2 under the current assumptions. We will assume this fact in what follows. We can therefore represent each group element in G (or H) by a word in $(A - \{id\})^*$.

Definition 3.1. Let $\widehat{\ell} : A - \{id\} \rightarrow \mathbb{R}^+$ (where \mathbb{R}^+ is the set of positive real numbers). The assignment $\widehat{\ell}$ determines two functions from $(A - \{id\})^*$ to $\mathbb{R}^+ \cup \{0\}$. For $w = a_1 \dots a_n$, ($a_i \in A - \{id\}$),

- (1) we set $|w| = \sum_{i=1}^n \widehat{\ell}(a_i)$.
- (2) we set $\ell(w) = \min\{|v| \mid v = w \text{ in } G(A)\}$.

We write C in place of $\ell(t)$.

Definition 3.2. Let $s \in S$. We say that $(s_{|0}, s_{|1})$ is the *first-line string production* of s . More generally, if $w \in (A - \{id\})^*$, we replace each letter $\widehat{s} \in S$ of w with the pair $(\widehat{s}_{|0}, \widehat{s}_{|1})$, and then collect all powers of the active state t at the end of the word using the relation $t(a, b) = (b, a)t$. Finally, we multiply all of the pairs together coordinate-wise, without any cancellation or application of relations from $G(A)$ (i.e., the multiplication takes place in $(A - \{id\})^*$). The resulting expression has the form $(\widetilde{w}_1, \widetilde{w}_2)(t)$ for some words $\widetilde{w}_1, \widetilde{w}_2 \in (A - \{id\})^*$; it is called the *first-line string production* of w . (Note that, here and in what follows, the t in $(\widetilde{w}_1, \widetilde{w}_2)(t)$ appears between parentheses because it may be present or not.)

If $w \in (A - \{id\})^*$, then we let $(\widetilde{w}_1, \widetilde{w}_2)(t)$ denote the first-line string production.

Example 3.3. We consider the automaton A from Example 2.3, and find the first-line string production of $w = tabtbabt$. We find

$$\begin{aligned} tabtbabt &= t(1, t)(b, a)t(b, a)(1, t)(b, a)t \\ &= (tabb, bata)t, \end{aligned}$$

which is the first-line string production of w .

Definition 3.4. For each element h in the group $H = \langle S \rangle$ we fix a word $w_h \in S^*$ such that

- (1) $w_h = h$ in G ;
- (2) If $\widehat{w} \in S^*$ is equal to h in G , then $|w_h| \leq |\widehat{w}|$.

(In other words, w_h is a representative for $h \in G$ that uses only letters from S , and such that $|\cdot| : S^* \rightarrow \mathbb{R}^+ \cup \{0\}$ is at a minimum over all such representatives.)

We let T denote the set of choices for w_h , where h ranges over all of H .

Let $w = (t^{\alpha_0}) w_1 t^{\alpha_1} w_2 t^{\alpha_2} \dots t^{\alpha_{m-1}} w_m (t^{\alpha_m})$ be an arbitrary word in $(A - \{id\})^*$, where each $w_i \in S^*$ and each $\alpha_i \geq 1$. (The parenthesized terms may be present or not.) We let $r(w)$ be the result of replacing each subword w_i with its representative in T . The latter assignment determines a function $r : (A - \{id\})^* \rightarrow (A - \{id\})^*$.

A word $w \in (A - \{id\})^*$ is in *weak reduced form* if it is in the range of r .

Note 3.5. In other words, we put an arbitrary string $w \in (A - \{id\})^*$ into weak reduced form by replacing each maximal string from S^* by its representative from T , without cancelling any ts . Thus, it is perfectly acceptable for a substring of the form t^n ($n \geq 2$) to appear in $r(w)$.

Note 3.6. It is easy to check that $r : (A - \{id\})^* \rightarrow (A - \{id\})^*$ has the property $r(r(u)r(v)) = r(uv)$, where the multiplication takes place in the free semigroup $(A - \{id\})^*$.

Definition 3.7. Let $w \in (A - \{id\})^*$. Let $(\widetilde{w}_0, \widetilde{w}_1)(t)$ be the first-line string production of w . The *first-line production* of w is $(r(\widetilde{w}_0), r(\widetilde{w}_1))(t)$.

We write $(w_0, w_1)(t)$ for the first-line production of w .

Suppose that $w = (t^{\alpha_0})w_1t^{\alpha_1}w_2t^{\alpha_2} \dots w_mt^{\alpha_m}$ is in weak reduced form. We set

$$|w|_* = (C) + (m-1)C + (C) + \sum_{i=1}^m |w_m|,$$

where the parenthesized terms will be absent if the corresponding factors t^{α_0} , t^{α_m} are absent in w .

Lemma 3.8. *For all $w \in (A - \{id\})^*$ in weak reduced form, we have $\ell(w) \leq |w|_* \leq |w|$.*

Proof. Let $w = (w_0)t^{\alpha_0}w_1t^{\alpha_1} \dots t^{\alpha_{m-1}}w_mt^{\alpha_m}(w_{m+1})$ be a word in weak reduced form. (Thus, $w_i \in T$, for all i .) We assume, for the sake of simplicity, that $w_0 = w_{m+1} = 1$.

$$\begin{aligned} |w| &= \sum_{i=0}^m \alpha_i C + \sum_{i=1}^m |w_i| \\ &\geq \sum_{i=0}^m C + \sum_{i=1}^m |w_i| \\ &= |w|_*. \end{aligned}$$

For each $i \in \{0, \dots, m\}$, let $\beta_i \in \{0, 1\}$ be the result of reducing α_i modulo 2. It follows that $t^{\alpha_0}w_1t^{\alpha_1} \dots t^{\alpha_{m-1}}w_mt^{\alpha_m} = t^{\beta_0}w_1t^{\beta_1} \dots t^{\beta_{m-1}}w_mt^{\beta_m}$ in the group $G(A)$. We have

$$\begin{aligned} |w|_* &\geq \sum_{i=0}^m \beta_i C + \sum_{i=1}^m |w_i| \\ &= |t^{\beta_0}w_1t^{\beta_1} \dots t^{\beta_{m-1}}w_mt^{\beta_m}| \\ &\geq \ell(w), \end{aligned}$$

where the final inequality follows from the definition of ℓ . \square

Definition 3.9. The length function $\ell : (A - \{id\})^* \rightarrow \mathbb{R}^+ \cup \{0\}$ is called *admissible* if $|t| + |w| \geq |w_0| + |w_1|$, for all $w \in T$, where $(w_0, w_1)(t)$ is the first-line production of w .

Lemma 3.10. *Let $w \in (A - \{id\})^*$ be in weak reduced form, and let $(w_0, w_1)(t)$ be the first-line production of w .*

- (1) $|w|_* + C \geq |w_0| + |w_1|$ if w contains fewer blocks of ts than blocks of letters from S ;
- (2) $|w|_* \geq |w_0| + |w_1|$ otherwise.

Proof. We prove (2) first. Let us assume that $w = t^{\alpha_1}\hat{w}_1t^{\alpha_2} \dots t^{\alpha_m}\hat{w}_m$ is in weak reduced form. We will confine our attention to this subcase, since the subcase $w = \hat{w}_1t^{\alpha_1} \dots t^{\alpha_{m-1}}\hat{w}_mt^{\alpha_m}$ is essentially similar, and the subcase $w = t^{\alpha_0}\hat{w}_1t^{\alpha_1} \dots t^{\alpha_{m-1}}\hat{w}_mt^{\alpha_m}$ is easier.

We have

$$\begin{aligned} |w|_* &= \sum_{i=1}^m (|t| + |\hat{w}_i|) \\ &\geq \sum_{i=1}^m (|(\hat{w}_i)_0| + |(\hat{w}_i)_1|). \end{aligned}$$

The desired conclusion $|w|_* \geq |w_0| + |w_1|$ now follows from the observation that there is a partition $\{P_0, P_1\}$ of $\{(\hat{w}_i)_j \mid i = 1, \dots, m; j = 0, 1\}$ such that each of the words w_k ($k = 0, 1$) in the first-line production $(w_0, w_1)(t)$ of w is obtained by multiplying the elements of P_k in some order (with each word in P_k appearing exactly once), and then applying r .

To prove (1), we note that w must have the form $w = \hat{w}_1 t^{\alpha_1} \dots t^{\alpha_{m-1}} \hat{w}_m t^{\alpha_m} \hat{w}_{m+1}$. We consider the word wt . By case (2),

$$|wt|_* \geq |(wt)_0| + |(wt)_1|,$$

but $|wt|_* = |w|_* + C$, and $(wt)_i = w_i$ for $i = 0, 1$. \square

3.2. Good and ϵ -good.

Definition 3.11. Let $w \in (A - \{id\})^*$ be in weak reduced form. A subword of w is called *protected* if it begins and ends with t , and contains at least one letter from S .

Definition 3.12. Write $\hat{w} \preccurlyeq w$ if there is some protected subword \tilde{w} of w such that \hat{w} is a protected subword of some word in the first-line production of \tilde{w} .

Lemma 3.13. Let w be a word representing an element of G_1 , and let u_1, \dots, u_n be protected subwords meeting (at most) in an initial or terminal block of ts . Let $\hat{u}_i \preccurlyeq u_i$ for $i = 1, \dots, n$. The first-line production of w contains occurrences of each of the \hat{u}_i . These occurrences meet (at most) in an initial or terminal block of ts and there are at least n distinct occurrences in all (one of each).

Proof. Consider the words $\hat{u}_1, \dots, \hat{u}_n$ obtained from u_1, \dots, u_n (respectively) by omitting the initial and terminal blocks of ts . We note that \hat{u}_i and u_i have the same first-line string productions (although they might produce the words in opposite coordinates). We write $w = v_1 \hat{u}_1 v_2 \hat{u}_2 \dots \hat{u}_n v_{n+1}$, where: (i) the v_i (for $1 \leq i \leq n$) end with blocks of ts (and may consist entirely of ts), and (ii) the v_i (for $2 \leq i \leq n+1$) begin with blocks of ts (and may consist entirely of ts), but are not trivial strings in either case. We form the first-line string production of w . Note that this is done word-by-word in the product $w = v_1 \hat{u}_1 v_2 \hat{u}_2 \dots \hat{u}_n v_{n+1}$. Each word v_i , \hat{u}_i contributes a string to both the left- and right-hand coordinates (although each may contribute the right-half of its first-line string production to the left word in the first-line string production of w , or vice versa).

Assume that

$$(\tilde{v}_1)_{i_1} (\tilde{u}_1)_{j_1} (\tilde{v}_2)_{i_2} \dots (\tilde{u}_n)_{j_n} (\tilde{v}_{n+1})_{i_{n+1}}$$

is the left-half of the first-line string production of $v_1 \hat{u}_1 \dots \hat{u}_n v_{n+1}$, where each string $(\tilde{v}_k)_{i_k}$, $(\tilde{u}_k)_{j_k}$ above is either the left or right half of the first-line string production of v_k , \hat{u}_k (respectively). Choose a word $(\tilde{u}_k)_{j_k}$ ($k \in \{1, \dots, n\}$). We assume without loss of generality that \hat{u}_k occurs as a protected subword in $(\hat{u}_k)_{j_k}$, where $(\hat{u}_k)_{j_k}$ denotes either the left- or right-half of the first-line production of \hat{u}_k . Since $\hat{u}_k \preccurlyeq \hat{u}_k$, there is some protected subword u such that $(\tilde{u}_k)_{j_k} = xuy$ and $r(u) = \hat{u}_k$. Thus $r((\tilde{u}_k)_{j_k}) = r(x)r(u)r(y)$. Since no ts are cancelled in an application of r , we get that $r(u) = \hat{u}_k$ occurs as a subword in w_0 . That is, \hat{u}_k occurs as a protected subword in w_0 . The lemma now follows. \square

Definition 3.14. Let w be a word in weak reduced form. A subword v of w is called *reducing* if $v \equiv tw_1 tw_2 tw_3 \dots tw_m$, where each $w_i \in T$, the word w_m is followed immediately by a t in w , $m \geq 1$, and $\ell(v_0) + \ell(v_1) < |v|_*$.

Definition 3.15. A word u is *good at depth m* if there are u_1, \dots, u_m such that

$$u \succ u_1 \succ \dots \succ u_m,$$

where u_m contains a reducing subword v .

We set

$$C_u = \frac{\ell(v_0) + \ell(v_1)}{|v|_*}$$

and

$$\sigma_u = |v|_*.$$

Definition 3.16. A word $w \in (A - \{id\})^*$ is *reduced* if $|w| = \ell(w)$.

Definition 3.17. Let $0 < \epsilon < 1$. A reduced word w of length N (i.e., $\ell(w) = N$) is ϵ -good with respect to the good word u if at least ϵN of its length is taken up by occurrences of u which meet in an initial or terminal t .

Definition 3.18. If $w \in (A - \{id\})^*$ represents an element of G_k and $\alpha \in \{0, 1\}^k$, then we let w_α denote the production of w in position α . For instance, w_0 denotes the left word in the first-line production of w . (Here we consider the “0” branch the left, and the “1” branch the right, half of the binary tree X^* .) If $k \geq 2$, then w_{01} would denote the right word in the first-line production of w_0 . We also say that w_{01} is in the *second-line production* of w , w_{011} is in the *third-line production*, and so forth.

Proposition 3.19. Let u be a good-at-depth- m word ($m \geq 0$). There are $0 < \theta < 1$ and $K > 0$ such that, for any $w \in (A - \{id\})^*$ satisfying

- (1) $|w| = \ell(w)$,
- (2) $\pi(w) \in G_{m+1}$, and
- (3) w is ϵ -good with respect to u ,

$$\sum_{\delta} \ell(w_{\delta}) \leq \theta \ell(w) + K.$$

The sum on the left is over all strings $\delta \in \{0, 1\}^{m+1}$.

Proof. Since occurrences of u contribute at least $\epsilon|w|$ to the total length of w , there must be at least $\epsilon(|w|/|u|)$ of them. We let β be the number of such occurrences; thus, $\beta \geq \epsilon(|w|/|u|)$. We let v, u_1, \dots, u_m be the words from Definition 3.15.

We choose θ so that $0 < \theta < 1$ and

$$\theta + \frac{\epsilon(1 - C_u)\sigma_u}{|u|} > 1.$$

It is clearly possible to find such θ , since the second term on the left is positive.

We first produce the m th line production of w . We have either:

- (1) $\sum_{\gamma} |w_{\gamma}|_* \leq \theta|w|$, for all w sufficiently large, or
- (2) $\sum_{\gamma} |w_{\gamma}|_* > \theta|w|$, for some sequence of words w such that $|w| \rightarrow \infty$.

The sums on the left are taken over all strings $\gamma \in \{0, 1\}^m$.

In the first case, we have

$$\begin{aligned} \theta|w| + 2^m C &\geq \sum_{\gamma} (|w_{\gamma}|_* + C) \\ &\geq \sum_{\gamma} |w_{\gamma 0}| + |w_{\gamma 1}| \\ &\geq \sum_{\delta} \ell(w_{\delta}), \end{aligned}$$

where the final sum is over all strings δ of length $m + 1$ in $\{0, 1\}$, and $(w_{\gamma 0}, w_{\gamma 1})$ is the first-line string production of w_{γ} . Thus there is nothing left to prove in this case.

Now assume that we are in the second case. For each $\gamma \in \{0, 1\}^m$, we let β_{γ} denote the number of occurrences of u_m in w_{γ} . By Lemma 3.13, we have $\sum_{\gamma} \beta_{\gamma} \geq \beta$.

The word w_{γ} has the form

$$w_{\gamma} = x_0 v x_1 v x_2 v \dots x_{\beta_{\gamma}-1} v x_{\beta_{\gamma}},$$

where each x_i is a word in weak reduced form, any one of which may be trivial, with the exception of $x_{\beta_{\gamma}}$, which must begin with a t if $\beta_{\gamma} > 0$. We note that each occurrence of v must be followed immediately by a t according to Definition 3.14. It follows that each x_i (for $i > 0$) begins with a t if it is not the null string. If some x_i ends with a block of ts , then we can combine this block with the word v following it (if any). The altered occurrence of v , v' , still contributes the same strings to the next line string production (though possibly in different places) and even satisfies $|v'|_* = |v|_*$, so we may ignore the additional powers of t for the sake of the following argument. One can now easily verify that each x_i ($i > 0$) contains at least as blocks of ts as blocks of letters from S .

We first claim that

$$|w_{\gamma}|_* + C + (C_u - 1)\beta_{\gamma}\sigma_u \geq \ell(w_{\gamma 0}) + \ell(w_{\gamma 1}),$$

for each $\gamma \in \{0, 1\}^m$. From the definition of $|\cdot|_*$ we see that

$$*) \quad |w_{\gamma}|_* = \sum_{i=0}^{\beta_{\gamma}} |x_i|_* + \sum_{i=1}^{\beta_{\gamma}} |v|_*.$$

It follows that

$$\begin{aligned} |w_{\gamma}|_* + C &= \sum_{i=1}^{\beta_{\gamma}} |x_i|_* + (|x_0|_* + C) + \beta_{\gamma}\sigma_u \\ &\geq \sum_{i=0}^{\beta_{\gamma}} [\ell((x_i)_0) + \ell((x_i)_1)] + \beta_{\gamma}C_u\sigma_u \\ &\geq \ell(w_{\gamma 0}) + \ell(w_{\gamma 1}). \end{aligned}$$

Here the first equality follows directly from $*)$. The expression on the second line is the $|\cdot|_*$ -length of the strings that result from taking the first-line productions of the words x_i and v , and reducing the results (i.e., substituting shortest-length strings) individually. The first inequality follows from Lemma 3.10. The final inequality is

now immediate. We also have

$$\begin{aligned} |w_\gamma|_* + C + \beta_\gamma C_u \sigma_u - \beta_\gamma \sigma_u &= \sum_{i=0}^{\beta_\gamma} |x_i|_* + C + \beta_\gamma C_u \sigma_u \\ &\geq \sum_{i=0}^{\beta_\gamma} [\ell((x_i)_0) + \ell((x_i)_1)] + \beta_\gamma C_u \sigma_u. \end{aligned}$$

The claim follows readily.

Applying the claim (and Lemma 3.10, repeatedly) we get

$$\begin{aligned} |w| + (1 + 2 + 2^2 + \dots + 2^m)C &\geq \sum_{\gamma} (|w|_* + C) \\ &\geq \sum_{\gamma} (|w|_* + C) + \beta \sigma_u (C_u - 1) \\ &\geq \sum_{\delta} \ell(w_\delta). \end{aligned}$$

At worst,

$$\sum_{\gamma} (|w|_* + C) + \beta \sigma_u (C_u - 1) > \theta |w|$$

for infinitely many w such that $|w| \rightarrow \infty$ (otherwise there is nothing left to prove).

We conclude that

$$\begin{aligned} |w| + (1 + 2 + 2^2 + \dots + 2^m)C &\geq \theta |w| + (1 - C_u) \beta \sigma_u \\ &\geq \theta |w| + (1 - C_u) \epsilon \frac{|w|}{|u|} \sigma_u \\ &= |w| \left(\theta + \frac{\sigma_u (1 - C_u) \epsilon}{|u|} \right). \end{aligned}$$

for all such w . It follows that

$$|w| + (1 + 2 + \dots + 2^m)C \geq D |w|,$$

where C , D , and m are constants, $D > 1$, and $|w| \rightarrow \infty$. This is a contradiction. \square

Definition 3.20. Let $0 < \epsilon < 1$. Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be such that u_i is a good-at-depth- m_i word, possibly for varying m_i . A reduced word w of length N is ϵ -good with respect to \mathcal{U} if at least ϵN of its length is taken up by occurrences of words from \mathcal{U} which meet (at most) in initial or terminal blocks of ts .

Theorem 3.21. Let $\mathcal{U} = \{u_1, \dots, u_n\}$ be as in Definition 3.20. Let $M = \max\{m_i\}$. There are $0 < \theta < 1$ and $K > 0$ such that, for all reduced words $w \in (A - \{id\})^*$ satisfying

- (1) $|w| = \ell(w)$,
- (2) $\pi(w) \in G_{M+1}$, and
- (3) w is ϵ -good with respect to \mathcal{U} ,

$$\sum_{\delta} \ell(w_\delta) \leq \theta \ell(w) + K.$$

The sum on the left is over all strings $\delta \in \{0, 1\}^{M+1}$.

Proof. Let w be an arbitrary word satisfying the given conditions. It follows that w is ϵ/n -good with respect to some word u_i , which is good at depth m_i . It follows from Proposition 3.19 that

$$\sum_{\delta_i} \ell(w_{\delta_i}) \leq \theta(u_i, \epsilon/n) \ell(w) + K(u_i, \epsilon/n),$$

where the sum on the left is over all $\delta_i \in \{0, 1\}^{m_i+1}$.

Lemma 3.10 implies that

$$\sum_{\delta} \ell(w_{\delta}) \leq \sum_{\delta_i} \ell(w_{\delta_i}) + (2^{m_i+1} + 2^{m_i+2} + \dots + 2^M) C.$$

We set $K_2(u_i, \epsilon/n) = (2^{m_i+1} + \dots + 2^M)C$. (In fact, K_2 depends only on u_i .)

We have

$$\sum_{\delta} \ell(w_{\delta}) \leq \theta(u_i, \epsilon/n) \ell(w) + K(u_i, \epsilon/n) + K_2(u_i, \epsilon/n).$$

We set $\theta(\mathcal{U}, \epsilon) = \max\{\theta(u_i, \epsilon/n)\}$, $K(\mathcal{U}, \epsilon) = \max\{K(u_i, \epsilon/n)\}$, and $K_2(\mathcal{U}, \epsilon) = \max\{K_2(u_i, \epsilon/n)\}$.

It now follows easily that

$$\sum_{\delta} \ell(w_{\delta}) \leq \theta(\mathcal{U}, \epsilon) \ell(w) + K(\mathcal{U}, \epsilon) + K_2(\mathcal{U}, \epsilon)$$

for all words w satisfying the hypotheses. \square

3.3. Subexponential Growth.

Definition 3.22. Let \mathcal{U} be a collection of good words. A reduced word w is \mathcal{U} -bad if no word of \mathcal{U} occurs as a subword of w .

Theorem 3.23. For $i = 1, \dots, n$, let u_i be a good-at-depth m_i word. Let $\mathcal{U} = \{u_1, \dots, u_n\}$. If there is some $M > 0$ such that, for all $L > 0$, there are at most M \mathcal{U} -bad words w of length L (i.e., $\ell(w) = L$), then $G(A)$ has subexponential growth.

Proof. For each $r > 0$ and all small $\epsilon > 0$, we estimate the number $b_{\epsilon}(r)$ of ϵ -bad words $w \in G_{N+1}$ of length ℓ precisely r . We will make the (harmless) assumption that each word has integral length. We have the following estimate:

$$b_{\epsilon}(r) \leq \binom{r+1}{\lfloor \epsilon r \rfloor + 1} M^{1+\lfloor \epsilon r \rfloor} |\mathcal{U}|^{\lfloor \epsilon r \rfloor}.$$

The right half of the inequality assumes that $\lfloor \epsilon r \rfloor$ good words appear in w (an absolute worst case). The factor $|\mathcal{U}|^{\lfloor \epsilon r \rfloor}$ counts the possible selections of those good words. The $\lfloor \epsilon r \rfloor$ good words divide w into $\lfloor \epsilon r \rfloor + 1$ pieces, all of which must be bad words. The binomial coefficient counts the number of solutions in non-negative integers to the inequality

$$\ell_1 + \dots + \ell_{\lfloor \epsilon r \rfloor + 1} \leq r - \lfloor \epsilon r \rfloor,$$

which over-counts the number of possible choices for the lengths of these bad words. Since the number of bad words of a given length is uniformly bounded by M , we find that there are at most $M^{1+\lfloor \epsilon r \rfloor}$ choices for these words.

Next, we consider the number $B_\epsilon(n)$ of reduced words $w \in G_{N+1}$ of length less than or equal to n .

$$\begin{aligned}
B_\epsilon(n) &= \sum_{r=0}^n b_\epsilon(r) \\
&\leq \sum_{r=0}^n \binom{r+1}{\lfloor \epsilon r \rfloor + 1} M^{1+\lfloor \epsilon r \rfloor} |\mathcal{U}|^{\lfloor \epsilon r \rfloor} \\
&\leq M^{1+\epsilon n} |\mathcal{U}|^{\epsilon n} \sum_{r=0}^n \binom{r+1}{\lfloor \epsilon r \rfloor + 1} \\
&\leq M^{1+\epsilon n} |\mathcal{U}|^{\epsilon n} (n+1) \binom{n+1}{\lfloor \epsilon n \rfloor + 1} \\
&\leq M^{1+\epsilon n} |\mathcal{U}|^{\epsilon n} (n+1) \frac{(n+1)^{\lfloor \epsilon n \rfloor + 1}}{(\lfloor \epsilon n \rfloor + 1)!}.
\end{aligned}$$

For large n , the latter quantity is approximately

$$M^{1+\epsilon n} |\mathcal{U}|^{\epsilon n} (n+1) e^{1+\lfloor \epsilon n \rfloor} \frac{1}{\sqrt{2\pi(\lfloor \epsilon n \rfloor + 1)}} \left(\frac{n+1}{\lfloor \epsilon n \rfloor + 1} \right)^{\lfloor \epsilon n \rfloor + 1},$$

by an application of Stirling's Formula to the quantity $(\lfloor \epsilon n \rfloor + 1)!$.

Now we suppose (for a contradiction) that G (thus, G_{N+1}) has exponential growth. Thus, there is some $\lambda > 1$ such that the ball of radius n in G_{N+1} has roughly λ^n elements. The estimate of $B_\epsilon(n)$ implies that we can choose ϵ sufficiently small that $B_\epsilon(n) \leq (\lambda_1)^n$, for some $1 < \lambda_1 < \lambda$. Thus, the proportion of ϵ -good elements in the ball of radius n to the total is at the least (roughly) $1 - \lambda_1/\lambda$. It follows that there is some positive proportion of reduced words $w \in G_{N+1}$ that satisfy the inequality in Theorem 3.21. In view of Proposition 10 from [2], we are done. \square

4. EXAMPLES OF GROUPS WITH INTERMEDIATE GROWTH

In practice, we will want to have one more type of next-line production, which is intermediate between the first-line string production and the first-line production.

Definition 4.1. Let $v = tw_1tw_2t \dots tw_m$, where each $w_i \in T$. Consider the following operation: Replace each w_i with its first-line production $((w_i)_0, (w_i)_1)$ and collect all powers of t at the end of the word, using the relation $t(a, b) = (b, a)t$. Multiply the pairs coordinate-by-coordinate with no cancellation (i.e., the multiplication occurs in $(A - \{id\})^*$). The result is called the *special production* of v .

Lemma 4.2. Let v be as in Definition 4.1, and suppose that $(\widehat{v}_0, \widehat{v}_1)(t)$ is its special production. If $|\widehat{v}_i| > \ell(\widehat{v}_i)$ for $i = 0$ or 1 , then $|v|_* > \ell(v_0) + \ell(v_1)$.

Proof. We note that the inequality $|v|_* \geq |\widehat{v}_0| + |\widehat{v}_1|$ follows immediately from Lemma 3.10, from which the desired conclusion follows by Note 3.6. \square

The groups associated to the automata with the kneading sequences $1(0^k)^\omega$, $11(0)^\omega$, and $0(011)^\omega$ will be proved to have subexponential growth in this section. In fact, all also have superpolynomial growth, since each group G is commensurable with $G \times G$, and this condition is known to imply superpolynomial growth (see [4], for instance).

4.1. **The Case of $1(0^k)^\omega$.** We consider the automata A_k ($k \geq 2$) with kneading sequence $1(0^k)^\omega$.

The graph Γ_{A_k} is depicted in Figure 2, which also indicates our convention for naming the states of A_k ($k \geq 2$). Namely, x_0 is the state adjacent to t (the unique active state), x_1 is the first state we encounter while tracing directed edges backwards from x_0 , x_2 is the second such state, and so forth.

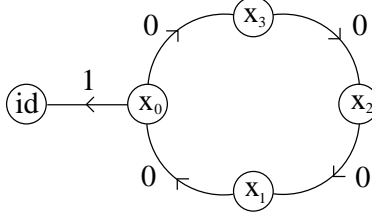


FIGURE 2. The graph Γ_A for the automaton with the kneading sequence $1(0^4)^\omega$.

Lemma 4.3. *Let x_0, x_1, \dots, x_{k-1} be the inactive states in the automaton A_k .*

$$\langle x_0, \dots, x_{k-1} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^k.$$

Proof. The action of x_i on a string $n_1 n_2 \dots n_m \in X^*$ can be described as follows. If $n_1 n_2 \dots n_m$ begins with a string of exactly j 0's ($j \geq 0$) followed by a 1, then $x_i \cdot n_1 \dots n_m = n_1 \dots n_m$ if $j \not\equiv i$ modulo k . If $j \equiv i$ modulo k , then $x_i \cdot n_1 n_2 \dots n_j n_{j+1} \hat{n}_{j+2} \dots$ where \hat{n}_{j+2} is 0 if n_{j+2} is 1, and 1 if n_{j+2} is 0. (If $n_1 n_2 \dots n_m$ contains no 1, or if $m = j + 1$, then $x_i \cdot n_1 \dots n_m = n_1 \dots n_m$.) It follows easily from this description that each x_i has order 2, and that any two elements of $\{x_0, \dots, x_{k-1}\}$ commute (since they have disjoint supports).

We claim that the words $x_{i_1} \dots x_{i_\alpha}$ ($\alpha \geq 0$) are all distinct, where i_1, \dots, i_α is an increasing sequence and $\{i_1, \dots, i_\alpha\} \subseteq \{0, \dots, k-1\}$. In fact $w := x_{i_1} x_{i_2} \dots x_{i_\alpha}$ has a description analogous to that of x_i : If $n_1 n_2 \dots n_m$ begins with a string of exactly j 0's ($j \geq 0$) followed by a 1, then $w \cdot n_1 \dots n_m = n_1 \dots n_m$ if $j \not\equiv i$ modulo k , for any $i \in \{i_1, \dots, i_\alpha\}$. If $j \equiv i$ modulo k for some such i , then $w \cdot n_1 n_2 \dots n_j n_{j+1} \hat{n}_{j+2} \dots$ where \hat{n}_{j+2} is 0 if n_{j+2} is 1, and 1 if n_{j+2} is 0. (If $n_1 n_2 \dots n_m$ contains no 1, or if $m = j + 1$, then $w \cdot n_1 \dots n_m = n_1 \dots n_m$.) It follows immediately that all such $x_{i_1} \dots x_{i_\alpha}$ are distinct, for distinct sequences i_1, \dots, i_α .

Thus, $\langle x_0, \dots, x_{k-1} \rangle$ contains at least 2^k elements, which implies that $\langle x_0, \dots, x_{k-1} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^k$, since our earlier argument shows that the former group is a quotient of the latter. \square

We assign weights to the states t, x_0, \dots, x_{k-1} as follows:

$$\begin{aligned} \widehat{\ell}(t) &= (k+2)^2 \\ \widehat{\ell}(x_i) &= (k+1-i) \quad (0 \leq i \leq k-1) \end{aligned}$$

We let $T = \{x_{i_1} \dots x_{i_\alpha} \mid \alpha \geq 0; 0 \leq i_1 < i_2 < \dots < i_\alpha \leq k-1\}$. It is not difficult to see that this choice T has the property required by Definition 3.4.

Lemma 4.4. *The length function $\ell : (A - \{id\})^* \rightarrow \mathbb{R}^+ \cup \{0\}$ induced by $\widehat{\ell}$ is admissible. In fact, $|t| + |w| > |w_0| + |w_1|$ for all $w \in T - \{x_0x_1x_2 \dots x_{k-1}\}$ (i.e., for all words in T except for the unique one using all k states x_0, \dots, x_{k-1}).*

Proof. Note that

$$\begin{aligned} x_0 &= (x_{k-1}, t); \\ x_i &= (x_{i-1}, 1) \quad (1 \leq i \leq k-1). \end{aligned}$$

We first consider the case in which w contains no occurrence of x_0 (i.e., $i_1 > 0$). In this case $|w_1| = 0$ and $w_0 \in T$. It follows that

$$|w_0| \leq \sum_{i=0}^{k-1} |x_i| = \frac{k^2 + 3k}{2} < |t|,$$

from which the strict inequality $|t| + |w| > |w_0| + |w_1|$ follows immediately.

Next, we suppose that w contains an occurrence of x_0 (i.e., $i_1 = 0$), but $w \neq x_0x_1 \dots x_{k-1}$. In this case $|w_1| = |t|$. Thus, to establish $|t| + |w| > |w_0| + |w_1|$, we want to show that $|w| > |w_0|$. We write $w = x_0x_{i_2} \dots x_{i_\alpha}$. Note that $\alpha < k$.

$$\begin{aligned} |w_0| &= |x_{k-1}x_{i_2-1}x_{i_3-1} \dots x_{i_\alpha-1}| \\ &= 2 + \sum_{\beta=2}^{\alpha} |x_{i_\beta-1}| \\ &= 2 + \sum_{\beta=2}^{\alpha} (k+1 - (i_\beta - 1)) \\ &= (\alpha + 1) + \sum_{\beta=2}^{\alpha} (k+1 - i_\beta) \\ &< (k+1) + \sum_{\beta=2}^{\alpha} (k+1 - i_\beta) \\ &= |w|, \end{aligned}$$

as required.

Finally, one easily sees that $|t| + |w| = |w_0| + |w_1|$ if $w = x_0x_1 \dots x_{k-1}$. \square

Theorem 4.5. *Each $G(A_k)$ ($k \geq 2$) has subexponential growth.*

Proof. We first nominate a set \mathcal{U} of good words. Let $\mathcal{U} = \{twt \mid w \in T - \{x_0x_1 \dots x_{k-1}\}\}$. Each word in this collection is good at depth 0 (the subword tw is a reducing word in each case, by Lemma 4.4).

By Theorem 3.23, to prove that the growth is subexponential it is enough to show that there is an $M > 0$ such that, for all $L > 0$, there are at most M \mathcal{U} -bad words of length L . Let us consider a \mathcal{U} -bad word of the form $tw_1tw_2t \dots tw_mt$ ($m > 0$). It is clear that each w_i must be $x_0x_1 \dots x_{k-1}$, so there is exactly one \mathcal{U} -bad word of this form for any m . A general bad word has the form $(w_0)tw_1t \dots tw_mt(w_{m+1})$, which shows that the number of such words is bounded above by a uniform constant that is independent of m . This easily implies the existence of the required M . \square

Note 4.6. All of the groups in this class have bad isotropy groups, so Theorem 2.21 does not guarantee that these groups are the iterated monodromy groups of complex polynomials.

4.2. The Case of $11(0)^\omega$. We now consider the group $G(A)$ of the kneading automaton A with kneading sequence $11(0)^\omega$. This automaton has already appeared as Example 2.3. The group $G(A)$ is generated by the automorphisms t , $a = (1, t)$, and $b = (b, a)$.

Lemma 4.7. *The group $\langle a, b \rangle$ is isomorphic to D_4 , the dihedral group of order 8.*

Proof. We first note that $a^2 = (1, t^2) = (1, 1) = 1$. We also have $b^2 = (b^2, a^2) = (b^2, 1)$. It follows easily by induction on the length m of a word $n_1 \dots n_m \in \{0, 1\}^*$ that b^2 also acts as the identity. Thus, $b^2 = 1$.

$$(ab)^4 = (b, at)^4 = (1, (at)^4).$$

Also,

$$(at)^4 = (atat)^2 = [(1, t)t(1, t)t]^2 = (t, t)^2 = 1.$$

It follows that $(ab)^4 = 1$.

We've now shown that $\langle a, b \rangle$ is a quotient of D_4 . We define a homomorphism $\phi : \langle a, b \rangle \rightarrow \langle a, t \rangle$ by $\phi(a) = t$; $\phi(b) = a$. (This homomorphism is restriction to the second coordinate, or restriction to the right branch of the tree $\{0, 1\}^*$.) Now one notes that $\langle a, t \rangle$ has order 8 as follows. We consider $00, 01, 10, 11 \in \{0, 1\}^*$. Relabel these vertices 1, 2, 3, and 4, respectively. It is straightforward to check that a acts as the permutation (34) and t acts as $(13)(24)$. It follows that $|\langle a, t \rangle| \geq 8$, so $|\langle a, b \rangle| \geq 8$. Thus, $|\langle a, b \rangle| = 8$, so $\langle a, b \rangle \cong D_4$. \square

We define $\widehat{\ell} : \{a, b, t\} \rightarrow \mathbb{R}^+$ by the rule $\widehat{\ell}(a) = \widehat{\ell}(b) = \widehat{\ell}(t) = 1$. We now fix representatives $w_h \in S^*$ for each $h \in \langle a, b \rangle$ as in Definition 3.4. Set

$$T = \{1, a, ab, aba, abab, b, ba, bab\}.$$

The first-line productions of the elements of T are (respectively) as follows:

$$(1, 1), (1, t), (b, ta), (b, tat), (1, tata), (b, a), (b, at), (1, ata).$$

One can easily check that the length function $\ell : (A - \{id\})^* \rightarrow \mathbb{R}^+ \cup \{0\}$ associated to $\widehat{\ell}$ is admissible. We summarize the relevant calculations in a table.

w	$ t + w $	$ w_0 + w_1 $
1	1	0
a	2	1
ab	3	3
aba	4	4
$abab$	5	4
b	2	2
ba	3	3
bab	4	3

We let α_n be the element of T of word-length n beginning with a . Let β_n denote the element of word-length n beginning with b . Thus, $T = \{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\}$.

Proposition 4.8. *The following families P_i of words in the generators $\{a, b, t\}$ are good, where each box \square represents an occurrence of a string from $\{\beta_1, \beta_2, \alpha_2\}$:*

$$P_0. \quad t\alpha_1 t, t\alpha_4 t, t\beta_3 t, t\alpha_3 t;$$

- P1. $t\Box t\beta_1t\Box t\beta_1t\Box;$
- P2. $t\beta_2t\Box t\beta_1t\Box t\beta_2t;$
- P3. $t\alpha_2t\Box t\beta_1t\Box t\beta_2t;$
- P4. $t\beta_2t\Box t\beta_1t\Box t\alpha_2t;$
- P5. $t\alpha_2t\Box t\beta_1t\Box t\alpha_2t;$
- P6. $t\beta_2t\Box t\beta_2t\Box t\beta_2t\Box t\alpha_2t\Box t\alpha_2t\Box t\alpha_2t,$ and
- P7. $t\alpha_2t\Box t\beta_2t.$

Note that the union $\mathcal{U} = \bigcup_{i=0}^7 P_i$ is also finite.

Proof. We consider each of the above families of words.

- P0. It is clear from the table that $t\alpha_1$, $t\alpha_4$, and $t\beta_3$ are reducing words in $t\alpha_1t$, $t\alpha_4t$, and $t\beta_3t$ (respectively), so the latter words are good at depth 0. The first-line production of $t\alpha_3t$ is (tat, b) . It follows that $tat = t\alpha_1t \preceq t\alpha_3t$, so $t\alpha_3t$ is good at depth 1.
- P1. The special production of $t\Box t\beta_1t\Box t\beta_1t\Box$ has the form $(_, babab)t$. Since $r(babab) = aba$ and $|aba| < |babab|$, each such word is a reducing word in $t\Box t\beta_1t\Box t\beta_1t\Box$ by Lemma 4.2. It follows that $t\Box t\beta_1t\Box t\beta_1t\Box$ is good at depth 0.
- P2. The first-line production of a word w in this family takes the form $(t\alpha_4t, _)$. It follows that $t\alpha_4t \preceq w$, for all words w in $P2$. Therefore, each such w is good at depth 1, since $t\alpha_4t \in P0$ is good at depth 0.
- P3. The special production of a word in this family takes the form $(tababat, _)$. It follows (as in P1) that each is good at depth 0; the reducing word omits the final t .
- P4. The first-line production of a word $w \in P4$ has the form $(atbabta, _)$. Thus, $t\beta_3t \preceq w$ for all words w in $P4$. Since $t\beta_3t \in P0$ is good at depth 0, we conclude that each $w \in P4$ is good at depth 1.
- P5. The first-line production of a word $w \in P5$ has the form $(tababta, _)$. It follows that $t\alpha_4t \preceq w$ for all words $w \in P5$. It now follows that each w is good at depth 1, since $t\alpha_4t$ is good at depth 0.
- P6. The first-line production of a word w in $P6$ has the form $(at\beta_2t\beta_2t\beta_1t\alpha_2t\alpha_2ta, _)$. It follows that $t\beta_2t\beta_2t\beta_1t\alpha_2t\alpha_2t \preceq w$. The former word is in family $P4$, so it is good at depth 1. Thus w is good at depth 2.
- P7. The first-line production of a word w in $P7$ has the form $(t\alpha_3t, _)$. Thus we have $t\alpha_3t \preceq w$, so w is good at depth 2 because $t\alpha_3t \in P0$ is good at depth 1.

The final statement is clear. \square

Theorem 4.9. *The group $G(A)$ of the automaton A with kneading sequence $11(0)^\omega$ has subexponential growth. The group $G(A)$ is also the iterated monodromy group of a complex post-critically finite quadratic polynomial.*

Proof. We prove the second statement first. In fact, by Theorem 2.21, it is enough to show that A is planar. This follows easily from the observation that $abtabt|_0 = bta$ and $abtabt|_1 = tab$.

We now turn to the first statement. Consider the collection of all reduced words that are \mathcal{U} -bad (where \mathcal{U} is as in Proposition 4.8). By Theorem 3.23, it is enough to show that there is $M > 0$ such that, for a given L , the number of \mathcal{U} -bad words of length L is less than M . It is clear that we may restrict our attention to large L .

We first consider bad words w that begin and end with t . Thus $w = tw_1tw_2t \dots tw_mt$ (for some large integer m). It is clear (from the description of $P0$) that $w_i \in \{\beta_1, \beta_2, \alpha_2\}$ for each i . We consider the possibilities for the sequence $w_1, w_3, w_5, \dots, w_{2k-1}$, where $2k-1$ is the largest odd number less than or equal to m .

First, suppose that $w_1 = \alpha_2$. It follows that $w_1, w_3, w_5, \dots, w_{2k-1}$ has one of the forms:

- (1) $\alpha_2, \alpha_2, \dots, \alpha_2, \alpha_2$;
- (2) $\alpha_2, \alpha_2, \dots, \alpha_2, \beta_1$;
- (3) $\alpha_2, \alpha_2, \dots, \alpha_2, \beta_1, \beta_1$ (if $2k-1 < m$).

Indeed, β_2 cannot follow α_2 , since that would create a subword from $P7$. If β_1 follows α_2 except in the last place (or in the second-to-last place, if $2k-1 < m$), then the next word in the sequence w_1, \dots, w_{2k-1} will be β_1, β_2 , or α_2 , which will create a subword from $P1, P3$, or $P5$ (respectively).

Now we consider the sequences w_1, \dots, w_{2k-1} such that $w_1 = \beta_2$ and some subsequent w_i is α_2 (for an odd subscript i). We claim that no occurrence of β_1 can appear between $w_1 (= \beta_2)$ and the earliest occurrence of α_2 . If $w_j = \beta_1$ is the earliest such occurrence, then $w_{j-2} = \beta_2$ and any choice of $w_{j+2} \in \{\beta_1, \beta_2, \alpha_2\}$ yields a subword from $P1, P2$, or $P4$ (respectively). This proves the claim. Now note that, once an α_2 occurs in $w_1, w_3, \dots, w_{2k-1}$, the remainder of the sequence takes one of the forms enumerated above, from the case in which $w_1 = \alpha_2$. In view of $P6$, the only possibilities are that $w_1, w_3, \dots, w_{2k-1}$ begins with 2 or fewer occurrences of β_2 followed by a sequence of one of the forms (1)-(3), or that w_1, \dots, w_{2k-1} begins with a long string of β_2 's, followed by a sequence of one of the forms (1)-(3) that contains 2 or fewer occurrences of α_2 .

Now we consider the sequences w_1, \dots, w_{2k-1} such that $w_1 = \beta_2$ and $w_1, \dots, w_{2k-1} \in \{\beta_1, \beta_2\}$. In view of $P1$, w_1, \dots, w_{2k-1} contains neither a subsequence of the form $\beta_1, \beta_1, \beta_2$, nor one of the form $\beta_1, \beta_1, \beta_1$. In view of $P2$, it cannot contain a subsequence of the form $\beta_2, \beta_1, \beta_2$. It follows that w_1, \dots, w_{2k-1} is a sequence of β_2 's ending with two or fewer occurrences of β_1 .

Now suppose that $w_1 = \beta_1$. If w_1, \dots, w_{2k-1} begins with three or more occurrences of β_1 , then we create a subword from $P1$, an impossibility. Thus, either w_3 or w_5 is in $\{\beta_2, \alpha_2\}$, and the remainder of the sequence takes one of the previously-discussed forms.

We have now completely described the possibilities for w_1, \dots, w_{2k-1} . Our discussion shows that w_1, \dots, w_{2k-1} is essentially a constant sequence of α_2 's or of β_2 's, with a small amount of variation possible at the beginning and end. It follows that the number of such sequences is bounded by a constant that is independent of k . A similar analysis establishes a similar form for the sequence w_2, w_4, \dots . It follows that the number of \mathcal{U} -bad words of the form $tw_1t \dots tw_mt$ is bounded, by a bound that is independent of m .

A general \mathcal{U} -bad word has the form $(w_0)tw_1t \dots tw_mt(w_{m+1})$, for some $w_0, w_{m+1} \in T$, and it follows immediately that the number of such words is uniformly bounded, regardless of m . Theorem 3.23 now implies that $G(A)$ has subexponential growth. \square

4.3. The Case of $0(011)^\omega$. We now consider the automaton A with kneading sequence $0(011)^\omega$. The graph Γ_A appears in Figure 3, which also indicates our convention for labeling the states.

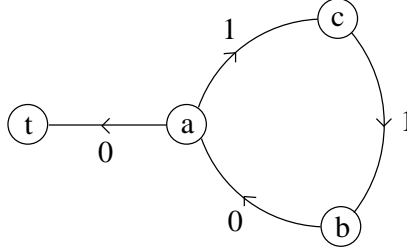


FIGURE 3. The graph Γ_A for the automaton A with kneading sequence $0(011)^\omega$.

Lemma 4.10. $\langle a, b, c \rangle \cong \langle a, b \rangle \times \langle c \rangle \cong D_8 \times \mathbb{Z}/2\mathbb{Z}$, where D_8 is the dihedral group of order 16.

Proof. First, we note that $a^2 = (t^2, c^2) = (1, c^2)$; $c^2 = (1, b^2)$; $b^2 = (a^2, 1)$. It follows from this that the automorphism a^2 has the following inductive definition: $a^2 \cdot n_1 n_2 \dots n_k = n_1 n_2 \dots n_k$ if $n_1 n_2 n_3 \neq 110$ (or if $k \leq 3$), and $a^2 \cdot 110 n_4 \dots n_k = 110 \cdot (a^2 \cdot n_4 \dots n_k)$. It follows directly that $|a| = 2$. Therefore, $|b| = 2$ and $|c| = 2$ by the above computations.

It follows immediately that $\langle a, b \rangle$ is a dihedral group. One can easily check that $(ta)^2 = (ct, tc)$, $(ta)^4 = ((b, b), (b, b))$, and (therefore) $(ta)^8 = 1$. Now

$$(ab)^n = ((ta)^n, c^n),$$

so $(ab)^8 = 1$. We've shown that $(ta)^4 \neq 1$, so $(ab)^4 \neq 1$, implying that $|ab| = 8$. Thus, $\langle a, b \rangle \cong D_8$.

Next we show that c commutes with a and b . The relation $bc = cb$ follows because c and b have disjoint support, and

$$ac = (t, cb) = (t, bc) = ca.$$

Finally, we show that $\langle c \rangle \cap \langle a, b \rangle = 1$. It is enough to show that $c \notin \langle a, b \rangle$. The elements of $\langle a, b \rangle$ all have the form $(_, c)$ or $(_, 1)$. Since $c = (1, b)$ and $b \neq c$, $c \notin \langle a, b \rangle$. It follows that $\langle c \rangle \cap \langle a, b \rangle = 1$, so $\langle a, b, c \rangle \cong \langle a, b \rangle \times \langle c \rangle$, as claimed. \square

We define $\widehat{\ell} : A - \{id\} \rightarrow \mathbb{R}^+$ by the rule

$$\widehat{\ell}(a) = 7; \widehat{\ell}(b) = 7; \widehat{\ell}(c) = 6 \widehat{\ell}(t) = 3.$$

We set

$$T = \{1, \alpha_1, \alpha_2, \dots, \alpha_8, \beta_1, \dots, \beta_7\} \cup \{c, c\alpha_1, \dots, c\alpha_8, c\beta_1, \dots, c\beta_7\}.$$

It is straightforward to check that T satisfies the conditions of Definition 3.4.

We list the words $\alpha_1, \alpha_2, \dots, \alpha_8, \beta_1, \dots, \beta_7$, their first-line productions, and the corresponding weights $|t| + |w|$ and $|w_0| + |w_1|$ in the table below. The first-line productions of the remaining words are obtained from the entries in the table simply by post-multiplying the second coordinates by b . Similarly, the weights $|t| + |w|$ and $|w_0| + |w_1|$ can be obtained by adding (respectively) 6 and 7 to the totals below. It follows easily that the length function ℓ is admissible.

w	(w_0, w_1)	$ t + w $	$ w_0 + w_1 $	w	(w_0, w_1)	$ t + w $	$ w_0 + w_1 $
α_1	(t, c)	10	9	β_1	$(a, 1)$	10	7
α_2	(ta, c)	17	16	β_2	(at, c)	17	16
α_3	$(tat, 1)$	24	13	β_3	(ata, c)	24	23
α_4	$(tata, 1)$	31	20	β_4	$(atat, 1)$	31	20
α_5	$(tatata, c)$	38	29	β_5	$(atata, 1)$	38	27
α_6	$(tatata, c)$	45	36	β_6	$(atatata, c)$	45	36
α_7	$(tatatat, 1)$	52	33	β_7	$(atatata, c)$	52	43
α_8	$((ta)^4, 1)$	59	40				

Proposition 4.11. *The following families of words in the generators $\{a, b, c, t\}$ are good, where each box \square represents an occurrence of a string from $\{\alpha_1, \alpha_2, c\beta_2, c\beta_3\}$:*

- P0.* twt , $w \in T - \{\alpha_1, \alpha_2, c\beta_2, c\beta_3\}$;
- P1.* $t\alpha_1 t \square t\alpha_1 t$;
- P2.* $t\alpha_1 t \square t\alpha_2 t$;
- P3.* $t \square t \square t\alpha_1 t \square t\beta_2 t \square t \square t$;
- P4.* $t \square t \square t\alpha_2 t \square t\alpha_1 t \square t \square t$;
- P5.* $t\alpha_2 t \square t\beta_2 t$;
- P6.* $t\alpha_2 t \square t\beta_3 t$;
- P7.* $t\beta_2 t \square t\alpha_1 t$;
- P8.* $t\beta_2 t \square t\alpha_2 t$;
- P9.* $t \square t \square t\beta_2 t \square t\beta_3 t \square t \square t$;
- P10.* $t \square t \square t\beta_3 t \square t\alpha_2 t \square t \square t$;
- P11.* $t\beta_3 t \square t\beta_2 t$;
- P12.* $t\beta_3 t \square t\beta_3 t$.

The union \mathcal{U} of the above families is finite.

Proof. We check that each family is made up of good words:

- P0. In this case, one can check that tw is a reducing word in each twt , so the members of this family are good at depth 0.
- P1. The first-line production of each word w in this family has the form $(_, tc\beta_1 t)$. Thus $tc\beta_1 t \preceq w$, so each w is good at depth 1, since $tc\beta_1 t \in P0$ is good at depth 0.
- P2. The first-line production of each word w in this family has the form $(_, tc\beta_1 ta)$. Thus $tc\beta_1 t \preceq w$, so each w is good at depth 1, since $tc\beta_1 t \in P0$ is good at depth 0.
- P3. Let $w \in P3$. We consider the first line production (w_0, w_1) of w . If the first \square in w is filled by either α_1 or $c\beta_2$, then w_1 contains a copy of $tc\beta_1 t$, and so w is good at depth 1. If the final \square in w is filled with either α_1 or α_2 , then w_1 again contains a copy of $tc\beta_1 t$, which makes w good at depth 1. In all other cases, w_1 contains a copy of $t\alpha_2 tc\beta_2 tc\beta_2 t$, which is good at depth 1 (see P5). Therefore, in this last case, w is good at depth 2.
- P4. Let $w \in P4$. We again consider w_1 in the first line production of w . If the first \square in w is filled by either α_1 or $c\beta_2$, then w_1 contains a copy of $tc\beta_1 t$, and so w is good at depth 1 (see P0). If the final \square in w is filled by either α_1 or α_2 , then w_1 contains a copy of $tc\beta_1 t$, so again w is good at depth

1. In every other case, w_1 contains a copy of $t\alpha_2 t\alpha_2 t\beta_2 t \in P5$. Since the latter word is good at depth 1, w is good at depth 2.
- P5. All words $w \in P5$ produce the word $w_1 = t\alpha_3 t$ on the first line. Since the latter word is good at depth 0 (see $P0$), we conclude that each $w \in P5$ is good at depth 1.
- P6. All words $w \in P6$ produce the word $w_1 = t\alpha_3 ta$. Since w_R contains the protected subword $t\alpha_3 t$, we conclude that each $w \in P6$ is good at depth 1.
- P7. All words $w \in P7$ produce the word $w_1 = at\beta_1 t$ on the first line. Since the protected subword $t\beta_1 t$ of w_1 is in $P0$, we conclude that w is good at depth 1.
- P8. All words $w \in P8$ produce $w_1 = at\beta_1 ta$ on the first line. It follows that w is good at depth 1.
- P9. Let $w \in P9$. If the first \square is filled by an occurrence of $c\alpha_2$ or $c\beta_3$, then w_1 contains an occurrence of $t\alpha_3 t$, which makes w good at depth 1. If the last \square is filled by an occurrence of $c\beta_2$ or $c\beta_3$, then w_1 again contains an occurrence of $t\alpha_3 t$, making w good at depth 1. In all other cases, w_1 contains an occurrence of $t\beta_2 t\beta_2 t\alpha_2 t \in P8$, which makes w good at depth 2.
- P10. Let $w \in P10$. If the first \square is filled by an occurrence of $c\alpha_2$ or $c\beta_3$, then w_1 contains an occurrence of $t\alpha_3 t$, which makes w good at depth 1. If the last \square is filled by an occurrence of $c\beta_2$ or $c\beta_3$, then w_1 again contains an occurrence of $t\alpha_3 t$, making w good at depth 1. In all other cases, w_1 contains an occurrence of $t\beta_2 t\alpha_2 t\alpha_2 t \in P8$, which makes w good at depth 2.
- P11. All words w in this family produce $w_1 = at\alpha_3 t$ on the first line. It follows that all words $w \in P11$ are good at depth 1, since $t\alpha_3 t \in P0$ is good at depth 0.
- P12. All words w in this family produce $w_1 = at\alpha_3 ta$ on the first line. It follows that all words $w \in P12$ are good at depth 1, since $t\alpha_3 t \in P0$ is good at depth 0.

Finally, it is clear that each of these families is finite, so their union \mathcal{U} is finite. \square

Theorem 4.12. *The group $G(A)$ determined by the automaton A with kneading sequence $0(011)^\omega$ has subexponential growth. The group $G(A)$ is the iterated monodromy group of a complex post-critically finite quadratic polynomial.*

Proof. We prove the second statement first. By Theorem 2.21, it is enough to show that the automaton is planar (the other conditions being obvious). The planarity of A follows from the equalities $(tacbtacb)_{|0} = cbta$ and $(tacbtacb)_{|1} = tacb$, both valid in $(A - \{id\})^*$.

We turn to a proof of the first statement. It is sufficient to show that there is $M > 0$ such that, for any $L > 0$, there are at most M \mathcal{U} -bad reduced words of length exactly L .

We first consider reduced words of the form $tw_1 tw_2 t \dots tw_m t$. Indeed, it is sufficient to consider words of this form, since a general reduced word has the form $(w_0)tw_1 t \dots tw_m t(w_{m+1})$, and such a word is \mathcal{U} -bad if and only if $tw_1 t \dots tw_m t$ is. Thus, the total number of bad words of the form $(w_0)tw_1 t \dots tw_m t(w_{m+1})$ is a constant multiple of the number of bad words of the form $tw_1 t \dots tw_m t$.

As in the proof of Theorem 4.9, we will consider the sequences w_1, w_3, w_5, \dots and w_2, w_4, w_6, \dots . The description of subfamily $P0$ shows that each w_i ($i = 1, \dots, m$) must be taken from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$. The descriptions of the remaining subfamilies $P1 - P12$ show (essentially; see below) that certain words from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$ must not be followed by certain other words from $\{c\alpha_1, c\alpha_2, c\beta_2, c\beta_3\}$ in the sequences w_1, w_3, \dots and w_2, w_4, \dots . Thus, for instance, the description of $P1$ implies that $c\alpha_1$ cannot follow $c\alpha_1$ in w_1, w_3, \dots or w_2, w_4, \dots .

The exceptions are $P3, P4, P9$, and $P10$. These subfamilies also forbid one word from following another, except possibly at the immediate end or beginning of w . We can ignore this distinction for the sake of this argument, however, since these exceptions allow only a finite amount of variation at the end and beginning of w , and this variation will simply increase the uniform bound M .

With this understanding, we can make the following observations. In w_1, w_3, \dots and w_2, w_4, \dots

- (1) $c\alpha_1$ can be followed only by $c\beta_3$;
- (2) $c\alpha_2$ can be followed only by $c\alpha_2$;
- (3) $c\beta_2$ can be followed only by $c\beta_2$;
- (4) $c\beta_3$ can be followed only by $c\alpha_1$.

Thus, modulo the above considerations, the only possibilities for the sequences w_1, w_3, \dots and w_2, w_4, \dots are those that alternate between $c\alpha_1$ and $c\beta_3$ and constant sequences of either $c\alpha_2$'s or $c\beta_2$'s. Thus the number of possible sequences $tw_1t \dots tw_mt$ is bounded above, by a constant independent of m . The existence of the uniform bound M now follows easily, and Theorem 3.23 establishes that $G(A)$ has subexponential growth. \square

5. A GROUP $G(A)$ WITH NO ADMISSIBLE LENGTH FUNCTION

We consider the kneading automaton with kneading sequence $01(10)^\omega$. We label the active state t , and label the remaining states a, b , and c , in the order that they are encountered while tracing directed edges backward from the active state in the Moore diagram. Thus $a = (t, 1)$, $b = (c, a)$, and $c = (1, b)$. Our goal in this section is to sketch a proof that $G(A)$ has no admissible length function.

Proposition 5.1. *The group $G(A)$ admits no admissible length function.*

Proof. Choose an arbitrary $\widehat{\ell} : A - \{id\} \rightarrow \mathbb{R}^+$ and an arbitrary T satisfying the conditions from Definition 3.4.

It turns out that: (1) the word c must be in T ; (2) one of the words $cbacb$, $cbcab$ must be in T , and (3) one of the words $baba$, $abab$ must be in T . To prove this, it helps to use the homomorphism $\phi : \langle a, b, c \rangle \rightarrow (\mathbb{Z}/2\mathbb{Z})^3$, where $\phi(a) = (1, 0, 0)$, $\phi(b) = (0, 1, 0)$, and $\phi(c) = (0, 0, 1)$. One establishes that ϕ is well-defined as follows. The subgroup $N = \langle abab, bc bc \rangle$ is central in $\langle a, b, c \rangle$, any two of the generators a, b, c commute modulo N , and the set $\{1, a, b, c, ab, ac, bc, abc\}$ is a transversal for N in $\langle a, b, c \rangle$. The existence of ϕ now follows directly from the First Isomorphism Theorem. One proves (1), (2), and (3) by first arguing that every other representative w' of the word w in question must have at least as many occurrences of each letter as w , and then arguing that no other permutations of the letters of w can represent the same group element. We omit the details.

Suppose, without loss of generality, that $\{c, baba, cbacb\} \subseteq T$, and assume that the length function $\ell : (A - \{id\})^* \rightarrow \mathbb{R}^+ \cup \{0\}$ is admissible. The first-line production of $cbacb$ is $(ctc, baba)$, so

$$|t| + 2|c| + 2|b| + |a| \geq |t| + 2|a| + 2|b| + 2|c|,$$

from which we conclude that $|a| \leq 0$. This is a contradiction. \square

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